Transport and entropy production due to chaos or turbulence

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A projection-operator method is developed for the statistical-mechanical formulation of chaotic or turbulent transport, such as chaos-induced friction in a forced damped pendulum and turbulent viscosity in a turbulent fluid. Then the nonlinear deterministic equations of motion for these dynamical systems are transformed into linear stochastic equations with chaotic or turbulent fluctuating forces. This leads to a fluctuation-dissipation formula which relates the chaotic or turbulent transport coefficients to the time correlation of the fluctuating forces. Applying this theory to the forced damped pendulum, we explore the chaos-induced friction and the power spectra of chaotic orbits. Applying it to the fluid turbulence governed by the Navier-Stokes equation, we find that the turbulent viscosity in the inertial subrange depends on wave number k as $k^{-\beta}$ with $\beta=\frac{4}{3}+\frac{1}{2}|\mu_{2/3}|$, μ_q being the intermittency exponent of order q.

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I. INTRODUCTION

Recently much attention has been paid to the study of various spatiotemporal phenomena observed far from equilibrium [1–4]. Particularly, the self-organized formation of macroscopic structures and sustained limit cycle oscillations are their typical examples and are called dissipative structures [4]. They are observed not only in laboratory experiments but also in the geophysics scale. The emergence of dissipative structures are controlled by changing the nonequilibrium parameter, the amount of energy injection rate to the system under consideration. It is not clarified yet, however, what physical processes underlie the dissipative structures when chaos or turbulence exists.

For certain ranges of nonequilibrium parameter, systems show chaotic or turbulent behaviors [4–6]. Chaos or turbulence is a very general phenomenon. It exists in quite wide ranges of nonlinear dissipative systems far from equilibrium, and exhibits various transport processes such as chaos-induced friction in a forced damped pendulum and turbulent viscosity in a turbulent fluid [7–9]. Their transport coefficients are much larger than the corresponding molecular transport coefficients, since their mixing lengths are much larger than the molecular mixing lengths such as the mean free paths of molecules in gases.

The transport processes are accompanied by the dissipation of the macroscopic kinetic energy into molecular thermal motions. This energy dissipation due to chaos or turbulence also brings about an entropy production which is much larger than that due to the molecular transport.

One of recent trends in the study of turbulence is the dynamical systems approach [10,11]. Therefore, it would be an interesting question whether we can formulate the chaotic or turbulent transport and energy dissipation in terms of the evolution equations of the system by extending the statistical mechanics of nonequilibrium systems near thermal equilibrium.

Thus it turns out that, in order to understand various physical processes in chaos and turbulence, the following must be studied from the statistical-mechanical point of view.

(1) The statistical properties of chaotic orbits and their fluctuations, such as their power spectra and time-correlation functions, where the variables of the chaotic orbits are regarded as the stochastic processes [12].

(2) The statistical properties characterizing the geometrical phase-space structures of chaos, such as the fluctuation spectra of local dimension and local expansion rate of nearby orbits [4].

(3) The statistical properties of the chaotic or turbulent transport, such as the chaos-induced friction and the turbulent viscosity, and their energy dissipation.

(4) The stochastic approach to chaos and turbulence.

The main purpose of this paper is to develop a general scheme for the statistical-mechanical formulation of the chaotic or turbulent transport and the stochastic approach to chaos and turbulence. This will be done by extending the statistical-mechanical formulation of the molecular transport near thermal equilibrium in terms of the molecular fluctuating force, which is often known as the theory of generalized Brownian motions [13–15].

Then the projection-operator method will be applied to the nonlinear dissipative systems far from equilibrium, so that the nonlinear evolution equations such as the equations of motion for the forced damped pendulum and the Navier-Stokes equation for fluid flow are transformed into linear stochastic equations with nonlinear chaotic or turbulent fluctuating forces, which are, however, non-Markovian. This amounts to the renormalization of the molecular transport coefficients by the nonlinear interactions which cause chaos or turbulence [16,17]. This also gives a generalization of Iwayama-Okamoto's theory [18] which explores the eddy viscosity in a two-dimensional inviscid fluid.

Thus we shall obtain a general scheme for the formulation

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of the chaotic or turbulent transport in terms of chaotic or turbulent fluctuating forces. This will give a fluctuation-dissipation formula for the chaotic or turbulent transport in dissipative systems far from equilibrium. In order to show the structure of the formulation explicitly, we shall apply the formulation to the chaos-induced friction in a forced damped pendulum and the turbulent viscosity in a turbulent fluid.

This paper is organized as follows. In Sec. II, we summarize the main features of the evolution equations for dissipative systems, and introduce the long-time average of the rate of entropy production. In Sec. III, the evolution of the time correlation of macrovariables is shown to be governed by a master equation for the δ-function density of macrovariables. In Sec. IV, we develop a general scheme for renormalizing the molecular transport by the nonlinear interactions so as to obtain the chaotic or turbulent transport explicitly. In Sec. V, the power spectra of chaotic orbits and the entropy production due to chaos or turbulence are investigated.

As the application, in Sec. VI we take the forced damped pendulum and explore its chaos-induced friction coefficient, power spectra and entropy production. In Sec. VII, we take homogeneous turbulence in an incompressible fluid governed by the Navier-Stokes equation, and explore the turbulent viscosity in terms of a turbulent fluctuating force and its scaling exponent in the inertial subrange. In Sec. VIII, we add an external noise to a system whose dissipative term is nonlinear, and show how the foregoing scheme must be extended. Section IX is devoted to a brief summary.

II. EVOLUTION EQUATIONS FOR DISSIPATIVE SYSTEMS

Let us consider the dissipative dynamical systems. One example is the periodically forced pendulum whose evolution equations for angle q and angular velocity p take the form [4]

$$\begin{pmatrix} \dot{q} \\ \dot{p} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} p \\ -\sin q - \gamma^0 p + b\cos\phi \\ \omega^0 \end{pmatrix}, \tag{2.1}$$

where γ^0 is the molecular friction coefficient and b is the amplitude of the driving force with angular frequency ω^0 and phase $\phi = \omega^0 t + \phi_0$. The nonlinear term $\sin q$ causes chaos when b increases beyond a critical value. The average rate of volume contraction in the phase space (q,p,ϕ) is given by

$$\lambda^{(d)} = \frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p} + \frac{\partial \dot{\phi}}{\partial \phi} = -\gamma^0. \tag{2.2}$$

Another example is the Navier-Stokes equation for the Fourier components of local fluid velocity $\mathbf{u_k} \equiv (u_{x\mathbf{k}}, u_{y\mathbf{k}}, u_{z\mathbf{k}})$ with wave vector $\mathbf{k}(k<k_c \sim 10^3 \text{ cm}^{-1})$, which takes the form [7]

$$\dot{u}_{\alpha\mathbf{k}} = v_{\alpha\mathbf{k}}(u) - \nu^0 k^2 u_{\alpha\mathbf{k}} + K_{\alpha\mathbf{k}} \tag{2.3}$$

for incompressible fluids, where $v_{\alpha\mathbf{k}}(u)$ is the inertial term

$$v_{\alpha\mathbf{k}}(u) \equiv \sum_{\beta,\gamma} {\sum_{\mathbf{p}}}' V_{\alpha\beta\gamma}(\mathbf{k}) u_{\beta\mathbf{p}} u_{\gamma\mathbf{k}-\mathbf{p}}, \tag{2.4}$$

$$V_{\alpha\beta\gamma}(\mathbf{k}) \equiv -\frac{i}{2}\{k_\beta \Delta_{\alpha\gamma}(\mathbf{k}) + k_\gamma \Delta_{\alpha\beta}(\mathbf{k})\} \tag{2.5}$$

with $\Delta_{\alpha\gamma}(\mathbf{k}) \equiv \delta_{\alpha\gamma} - (k_\alpha k_\gamma / k^2)$. Here $\Sigma'_{\mathbf{p}}$ is the sum over $\mathbf{p}$'s with a cutoff $k_c(p<k_c)$, ν^0 is the kinematic molecular viscosity, and $K_{\alpha\mathbf{k}}$ is a steady external force. The inertial term $v_{\alpha\mathbf{k}}(u)$ represents the nonlinear interactions between hydrodynamic modes and causes turbulence when the Reynolds number increases beyond a critical value. The average rate of phase-space volume contraction is given by

$$\lambda^{(d)} = \sum_\alpha {\sum_{\mathbf{k}}}' \overline{\frac{\partial \dot{u}_{\alpha\mathbf{k}}}{\partial u_{\alpha\mathbf{k}}}} = -3 {\sum_{\mathbf{k}}}' \nu^0 k^2, \tag{2.6}$$

where the bar means the long-time average.

It turns out from Eqs. (2.1) and (2.3) that the evolution equations for a complete set of macrovariables $A(t) \equiv \{A_\ell(t)\}, (\ell = 1,2,\dots)$ consist of three terms

$$\dot{A}_\ell(t) = v_\ell(A(t)) + J^0_\ell(A(t)) + K_\ell(A(t)), \tag{2.7}$$

where $v_\ell(A)$ is a nonlinear reversible term which causes chaos or turbulence, $J^0_\ell(A)$ is a linear irreversible term

$$J^0_\ell(A) = -\sum_n \Gamma^0_{\ell n} A_n, \tag{2.8}$$

and $K_\ell(t) \equiv K_\ell(A(t))$ is a periodic external force with a frequency ω^0. The matrix $\Gamma^0_{\ell n}$ represents the average rate of energy dissipation due to the molecular transport, such as γ^0 of Eq. (2.1) and $\nu^0 k^2$ of Eq. (2.3).

Under time reversal $t \to -t$, $A_\ell(t) \to \epsilon_\ell A_\ell(t)$, $(\epsilon_\ell = +1$ or $-1)$, $\omega^0 \to -\omega^0$, we have

$$\dot{A}_\ell(t) \to -\epsilon_\ell \dot{A}_\ell(t), \quad v_\ell(A) \to -\epsilon_\ell v_\ell(A),$$
$$J^0_\ell(A) \to \epsilon_\ell J^0_\ell(A), \quad K_\ell(A) \to -\epsilon_\ell K_\ell(A). \tag{2.9}$$

Therefore, if we neglect the term $J^0_\ell(A)$, then the evolution equations (2.7) are invariant under time reversal. In other words, the terms $v_\ell(A)$ and $K_\ell(A)$ are reversible, whereas the term $J^0_\ell(A)$ is irreversible. This irreversible term brings about the phase-space volume contraction

$$\lambda^{(d)} = \sum_\ell \overline{\frac{\partial \dot{A}_\ell}{\partial A_\ell}} = \sum_\ell \frac{\partial J^0_\ell}{\partial A_\ell} = -\sum_\ell \Gamma^0_{\ell\ell}. \tag{2.10}$$

This irreversibility arises from the molecular transport such as γ^0 and ν^0, which brings about the dissipation of the macroscopic energy into microscopic thermal motions, i.e., the conversion of the macroscopic energy into heat. Its average rate of energy dissipation is given by $|\lambda^{(d)}| = \Sigma_\ell \Gamma^0_{\ell\ell}$.

The physical quantity which represents this energy dissipation is the entropy production which is given in the following manner. Let us take a statistical ensemble of systems in the phase space spanned by $\{A_\ell\}$, and denote their density by $w(A)$. Let us consider the time evolution of $w(A(t))$ along the orbit $A(t)$. The conservation of systems leads to the continuity equation $\partial w/\partial t = -\Sigma_\ell(\partial/\partial A_\ell) w\dot{A}_\ell$. Hence, along the orbit $A(t)$, we have

$$\frac{d}{dt}\ln w(A(t)) \equiv \frac{1}{w}\left[\frac{\partial w}{\partial t} + \sum_\ell \dot{A}_\ell \frac{\partial w}{\partial A_\ell}\right] = -\sum_\ell \frac{\partial \dot{A}_\ell}{\partial A_\ell}. \tag{2.11}$$

Then Boltzmann's entropy $S(A) = k_B \ln w(A)$ leads to the average rate of entropy production [19]

$$\overline{\dot{S}} = k_B\left|\sum_\ell \overline{\frac{\partial \dot{A}_\ell}{\partial A_\ell}}\right| = k_B\left|\sum_\ell \frac{\partial J^0_\ell}{\partial A_\ell}\right| = k_B \sum_\ell \Gamma^0_{\ell\ell}, \tag{2.12}$$

k_B being the Boltzmann constant. Therefore, $\overline{\dot{S}} = k_B|\lambda^{(d)}|$. It should be noted, however, that this does not contain the entropy production due to chaos or turbulence.

III. TIME CORRELATIONS OF MACROVARIABLES

Let us take the time-correlation functions

$$\overline{A_\ell(t)A^\dagger_m(0)} \equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T A_\ell(t+s)A^\dagger_m(s)ds, \tag{3.1}$$

a dagger indicating the Hermite conjugate. This two-body correlation is coupled with higher order correlations

$$\overline{A_{\ell_1}(t)A_{\ell_2}(t)A^\dagger_m(0)}, \quad \overline{A_{\ell_1}(t)A_{\ell_2}(t)A_{\ell_3}(t)A^\dagger_m(0)}, \quad \ldots,$$

via the nonlinear term $v_\ell(A)$, as can be seen by substituting Eq. (2.7) into $\overline{\dot{A}_\ell(t)A^\dagger_m(0)}$.

We have

$$A_{\ell_1}(t)\cdots A_{\ell_n}(t) = \int a_{\ell_1}\cdots a_{\ell_n} g_a(t)da \tag{3.2}$$

in terms of the δ-function density [17,20]

$$g_a(t) \equiv \delta[A(t)-a] \equiv \prod_\ell \delta[A_\ell(t)-a_\ell]. \tag{3.3}$$

Hence the higher order correlations can be written as

$$\overline{A_{\ell_1}(t)\cdots A_{\ell_n}(t)A^\dagger_m(0)} = \int a_{\ell_1}\cdots a_{\ell_n}\overline{g_a(t)A^\dagger_m(0)}da. \tag{3.4}$$

Therefore, in order to avoid the difficulty of the infinite chain of higher order correlations, let us consider the time evolution of $g_a(t)$.

The time evolution of $g_a(t)$ is given by

$$\frac{\partial}{\partial t}g_a(t) = -\sum_\ell \frac{\partial}{\partial a_\ell}\{\dot{A}_\ell(t)\delta[A(t)-a]\} = Mg_a(t), \tag{3.5}$$

where substituting Eq. (2.7) gives the operator

$$M \equiv -\sum_\ell \frac{\partial}{\partial a_\ell}\{v_\ell(a) + J^0_\ell(a) + K_\ell(a)\}. \tag{3.6}$$

This is integrated to give

$$g_a(t) = \exp[tM]g_a(0). \tag{3.7}$$

It is convenient to introduce the operator

$$\Lambda \equiv \sum_\ell \{v_\ell(a) + J^0_\ell(a) + K_\ell(a)\}\frac{\partial}{\partial a_\ell}, \tag{3.8}$$

which is conjugate to M. Then Eq. (3.2) gives

$$A_\ell(t) = \int a_\ell\{e^{tM}g_a(0)\}da = \int a_\ell(t)g_a(0)da, \tag{3.9}$$

where the partial integration gives [17]

$$a_\ell(t) \equiv \exp[t\Lambda]a_\ell. \tag{3.10}$$

Since $A_\ell(t) = [a_\ell(t)]_{a=A(0)}$ from Eq. (3.9), the time evolution of $a_\ell(t)$ coincides with that of $A_\ell(t)$. Therefore, we have $f(A(t)) = [f(a(t))]_{a=A(0)}$ for any function of $A(t)$. We also have

$$f(A(t)) = \int f(a)g_a(t)da = [e^{t\Lambda}f(a)]_{a=A(0)}. \tag{3.11}$$

This is compared with the above to give

$$e^{t\Lambda}f(a) = f(e^{t\Lambda}a) = f(a(t)). \tag{3.12}$$

For the Hamiltonian systems, we have $\Lambda = iL = -M$, L being the Liouville operator.

Substituting Eq. (3.9) into Eq. (3.1) gives

$$\begin{aligned}\overline{A_\ell(t)A^\dagger_m(0)} &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\int a_\ell(t)g_a(s)A^\dagger_m(s)dads\\ &= \langle a_\ell(t)a^\dagger_m\rangle,\end{aligned} \tag{3.13}$$

where we have defined the ensemble average

$$\langle\cdots\rangle \equiv \int P_*(a)\cdots da \tag{3.14}$$

in terms of the probability density

$$P_*(a) \equiv \overline{g_a(s)} \equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T g_a(s)ds. \tag{3.15}$$

Taking the long-time average of Eq. (3.5), we get

$$MP_*(a) = 0. \tag{3.16}$$

In the following, $A_\ell(t)$ and $a_\ell(t)$ are set so as to be $\overline{A_\ell(t)} = \langle a_\ell(t)\rangle = 0$. Then the time-correlation functions (3.13) would decay to zero in time as

$$\langle a_\ell(t) a_m^\dagger \rangle \to 0 \quad \text{for} \quad t \to \infty, \tag{3.17}$$

representing the mixing of chaotic orbits in chaotic or turbulent systems.

IV. LINEAR STOCHASTIC EQUATIONS

The evolution equations (2.7) do not exhibit the energy dissipation due to chaos or turbulence explicitly, even though the nonlinear term $v_\ell(a)$ causes chaos or turbulence. As will be shown in this section, however, the nonlinear term $v_\ell(a)$ can be transformed into the sum of a linear transport term due to chaos or turbulence and a chaotic or turbulent fluctuating force.

Let us take the Hilbert space for functions of $a=\{a_\ell\}$, where the inner product of $f_1(a)$ and $f_2(a)$ is defined by the average $\langle f_1(a) f_2^\dagger(a) \rangle$, $\langle \cdots \rangle$ implying the average (3.14). Then let us introduce the projection of a vector $f(a)$ onto the vector a [13]:

$$\mathcal{P} f(a) = \sum_m \sum_n \langle f(a) a_m^\dagger \rangle [\langle a a^\dagger \rangle^{-1}]_{mn} a_n, \tag{4.1}$$

where $\langle a a^\dagger \rangle^{-1}$ is the inverse of the square matrix $\{\langle a_\ell a_m^\dagger \rangle\}$. Namely, $\mathcal{P}$ is the projection operator which extracts the linear part of the operand.

Equation (3.8) gives $\Lambda a_\ell = v_\ell(a) + J_\ell^0(a) + K_\ell(a)$, so that the time evolution of Eq. (3.10) gives

$$\dot{a}_\ell(t) = e^{t\Lambda} v_\ell(a) - \sum_n \Gamma_{\ell n}^0 a_n(t) + K_\ell(a(t)), \tag{4.2}$$

where Eq. (2.8) has been used. Using $\mathcal{Q} = 1 - \mathcal{P}$, we have $v_\ell(a) = \mathcal{P} v_\ell(a) + \mathcal{Q} v_\ell(a)$. Then the first term of Eq. (4.2) can be written as

$$e^{t\Lambda} v_\ell(a) = \sum_n i\Omega_{\ell n}^0 a_n(t) + e^{t\Lambda} \mathcal{Q} v_\ell(a), \tag{4.3}$$

where $i\Omega^0$ is the frequency matrix

$$i\Omega_{\ell n}^0 \equiv \sum_m \langle v_\ell(a) a_m^\dagger \rangle [\langle a a^\dagger \rangle^{-1}]_{mn}. \tag{4.4}$$

Putting $\Lambda = \mathcal{Q}\Lambda + \mathcal{P}\Lambda$, we have

$$e^{t\Lambda} = e^{t\mathcal{Q}\Lambda} + \int_0^t e^{(t-s)\Lambda} \mathcal{P}\Lambda e^{s\mathcal{Q}\Lambda} ds, \tag{4.5}$$

which can be proved by checking that both sides are unity at $t=0$ and their derivatives are $\Lambda e^{t\Lambda}$. Substituting Eq. (4.5) into the second term of Eq. (4.3) and then substituting it into Eq. (4.2) leads to

$$\dot{a}_\ell(t) = \sum_n (i\Omega_{\ell n}^0 - \Gamma_{\ell n}^0) a_n(t) - \sum_n \int_0^t \Gamma'_{\ell n}(s) a_n(t-s) ds + r_\ell(a,t) + K_\ell(a(t)), \tag{4.6}$$

where we have defined

$$r_\ell(a,t) \equiv e^{t\mathcal{Q}\Lambda} \mathcal{Q} v_\ell(a), \tag{4.7}$$

$$\Gamma'_{\ell n}(s) \equiv -\sum_m \langle \{\Lambda r_\ell(a,s)\} a_m^\dagger \rangle [\langle a a^\dagger \rangle^{-1}]_{mn}. \tag{4.8}$$

Thus, first extracting the linear term $e^{t\Lambda} \mathcal{P} v_\ell(a)$ from $e^{t\Lambda} v_\ell(a)$ as in Eq. (4.3), and then extracting the linear term generated by the time evolution of $e^{t\Lambda} \mathcal{Q} v_\ell(a)$ as in the second term of Eq. (4.5), we are left with the $r_\ell(a,t)$ of Eq. (4.7), which is completely nonlinear. This $r_\ell(a,t)$ is called the chaotic or turbulent fluctuating force. The memory function $\Gamma'_{\ell n}(s)$ decays in a macroscopic time τ_r, representing the loss of memory with respect to the initial conditions, and brings about the energy dissipation due to the chaotic or turbulent transport. Thus we obtain the linear stochastic equations (4.6), where the frequency matrix $i\Omega^0$ gives a coherent oscillation in the chaotic motion and the memory function $\Gamma'(t)$ describes the mixing in the chaotic motion.

The evolution operator of $r_\ell(a,t)$ is $\mathcal{Q}\Lambda$ in contrast to Eq. (3.10). Since $\mathcal{P}\mathcal{Q} = \mathcal{P} - \mathcal{P}^2 = 0$, therefore, we have $\mathcal{P} r_\ell(a,t) = 0$, i.e.,

$$\langle r_\ell(a,t) a_m^\dagger \rangle = 0, \quad (t \geqslant 0). \tag{4.9}$$

This is the most important feature of the fluctuating force $r_\ell(a,t)$. Hence, multiplying Eq. (4.6) by $a_m^\dagger$ and taking the average $\langle \cdots \rangle$, we obtain

$$\langle \dot{a}_\ell(t) a_m^\dagger \rangle = \sum_n (i\Omega_{\ell n}^0 - \Gamma_{\ell n}^0) \langle a_n(t) a_m^\dagger \rangle - \sum_n \int_0^t \Gamma'_{\ell n}(s) \times \langle a_n(t-s) a_m^\dagger \rangle ds. \tag{4.10}$$

This is the linear transport equations for the time-correlation matrix $\langle a(t) a^\dagger \rangle$, so that its Laplace transform takes the form [13]

$$\Xi(\omega) \equiv \int_0^\infty e^{-i\omega t} \langle a(t) a^\dagger \rangle \cdot \langle a a^\dagger \rangle^{-1} dt, \tag{4.11}$$

$$= \frac{1}{i(\omega - \Omega^0) + \Gamma^0 + \Gamma'(\omega)}, \tag{4.12}$$

where $\Gamma'(\omega)$ is the Laplace transform of the memory function matrix $\Gamma'(t)$ and represents the energy dissipation rate due to the chaotic or turbulent transport. Thus the linear stochastic equations (4.6) are very useful for formulating the transport and energy dissipation due to chaos or turbulence. Indeed, an evolution equation of this type has been used as the basis for the statistical-mechanical formulation of transport and thermal fluctuations near thermal equilibrium [13–17]. In the following it will turn out that the linear stochastic equations (4.6) also give the basis for a stochastic description of chaos and turbulence in dissipative systems far from equilibrium.

The $r_\ell(a,s)$ factor of Eq. (4.8) becomes

$$\langle\{\Lambda r_\ell(a,s)\}a_m^\dagger\rangle=\int r_\ell(a,s)M\{a_m^\dagger P_*(a)\}da,$$

$$=-\langle r_\ell(a,s)\{\Lambda a_m^\dagger\}\rangle, \tag{4.13}$$

where use has been made of $MP_*(a)=0$. Since $\langle r_\ell(a,s)\mathcal{P}\{\Lambda a_m^\dagger\}\rangle=0$, $\mathcal{Q}\Lambda a_m^\dagger=\mathcal{Q}v_m^\dagger=r_m^\dagger(a,0)$, we have

$$\langle\{\Lambda r_\ell(a,s)\}a_m^\dagger\rangle=-\langle r_\ell(a,s)r_m^\dagger(a,0)\rangle. \tag{4.14}$$

Substituting this into Eq. (4.8) gives, in the matrix form

$$\Gamma'(t)=\langle r(a,t)r^\dagger(a,0)\rangle\cdot\langle aa^\dagger\rangle^{-1}. \tag{4.15}$$

Then the dissipation rate $\Gamma'(\omega)$ takes the form

$$\Gamma'(\omega)=\int_0^\infty e^{-i\omega t}\langle r(a,t)r^\dagger(a,0)\rangle\cdot\langle aa^\dagger\rangle^{-1}dt. \tag{4.16}$$

The physical experiments such as the light scatterings observe Eq. (4.11) and explore the dissipation rate $\Gamma^0+\Gamma'(\omega)$. Here Eq. (4.16) gives the fluctuation-dissipation formula which relates the dissipation rate $\Gamma'(\omega)$ to the time-correlation function of the chaotic or turbulent fluctuating force $r(a,t)$.

Thus, in contrast to the evolution equations (2.7), the evolution equations (4.6) contain the memory function $\Gamma'(t)$ for the chaotic or turbulent transport and the corresponding fluctuating force $r(a,t)$ explicitly. This has been derived by renormalizing the molecular dissipation rate Γ^0 by the nonlinear interactions $\mathcal{Q}v_\ell(a)$ so as to give the chaotic or turbulent dissipation rate $\Gamma'(\omega)$ explicitly [17]. Therefore, Eq. (4.6) and $\dot{A}_\ell(t)=[\dot{a}_\ell(t)]_{a=A(0)}$ may be called the renormalized evolution equations. In this paper, however, these are referred to as the linear stochastic equations which give a stochastic description of chaos and turbulence.

V. POWER SPECTRA AND ENTROPY PRODUCTION

Let us suppose that a chaotic orbit $a(t)$ is observed over a long time interval $0\leqslant t\leqslant T$ with $T\gg\tau_M(\geqslant\tau_r)$, where τ_M and τ_r are the decay times of $\langle a(t)a^\dagger\rangle$ and $\langle r(a,t)r^\dagger(a,0)\rangle$, respectively. Now let us expand $a_\ell(t)$ in a Fourier series as

$$a_\ell(t)=\sum_j{}' a_\ell(\omega_j)e^{i\omega_j t}, \tag{5.1}$$

where the frequencies are

$$\omega_j=\frac{2\pi j}{T}, \quad (j=0,\pm1,\pm2,\dots) \tag{5.2}$$

and Σ_j' is the sum over ω_j's with a cutoff ω_c satisfying $1/\tau_M\leqslant\omega_c\ll1/\tau_m$ for a microscopic time scale τ_m. Then the linear stochastic equations (4.6) lead, for $t\gg\tau_M$, to

$$i\omega a_\ell(\omega)=\sum_n\{i\Omega^0_{\ell n}-\Gamma^0_{\ell n}-\Gamma'_{\ell n}(\omega)\}a_n(\omega)+r_\ell(a,\omega), \tag{5.3}$$

where $\omega\neq\pm\omega^0$, ω^0 being the frequency of $K_\ell(t)$. The power spectrum of $a_\ell(t)$ is given by [12]

$$I_\ell(\omega)\equiv\lim_{T\to\infty}\frac{T}{2\pi}\langle|a_\ell(\omega)|^2\rangle$$

$$=\frac{1}{2\pi}\left\{\sum_m \Xi_{\ell m}(\omega)\langle a_m a_\ell^\dagger\rangle+\text{c.c.}\right\}. \tag{5.4}$$

Let us consider the low frequency components with $|\omega|\ll1/\tau_r(\leqslant\omega_c)$, so that we have $\tau_M\gg\tau_r$ and the dissipation rate $\Gamma'(\omega)$ may be regarded as a constant $\Gamma'\equiv\Gamma'(\omega=0)$. Then, introducing the coarse graining of macrovariables in time

$$A_\ell(t)\equiv\sum_j{}'' A_\ell(\omega_j)e^{i\omega_j t} \tag{5.5}$$

with $A_\ell(\omega_j)\equiv[a_\ell(\omega_j)]_{a=A(0)}$ and Σ_j'' being the sum over ω_j's with a cutoff $\omega_L(\ll1/\tau_r)$, we obtain

$$\dot{A}_\ell(t)=\sum_n i\Omega^0_{\ell n}A_n(t)+J_\ell(A(t))+r_\ell(t)+K_\ell(A(t)), \tag{5.6}$$

where we have defined the dissipative flux

$$J_\ell(A)\equiv-\sum_n\{\Gamma^0_{\ell n}+\Gamma'_{\ell n}\}A_n \tag{5.7}$$

and the fluctuating force $r_\ell(t)\equiv[r_\ell(a,t)]_{a=A(0)}$. It should be noted that the memory term of Eq. (4.6) is reversible under time reversal (2.9), i.e.,

$$\int_0^t\Gamma'_{\ell n}(s)a_n(t-s)ds\to\epsilon_n\int_0^{-t}\Gamma'_{\ell n}(s)a_n(t+s)ds,$$

$$=-\epsilon_n\int_0^t\Gamma'_{\ell n}(\tau)a_n(t-\tau)d\tau$$

with $\Gamma'_{\ell n}(-\tau)=\Gamma'_{\ell n}(\tau)$, whereas $\Gamma'_{\ell n}a_n(t)$ is irreversible, i.e., $\Gamma'_{\ell n}a_n(t)\to\epsilon_n\Gamma'_{\ell n}a_n(t)$. This irreversibility arises from the coarse graining in time introduced in the above. Therefore, the dissipative flux (5.7) is irreversible and contributes to the entropy production. Thus, instead of Eq. (2.12), we obtain

$$\overline{\dot{S}}=k_B\left|\sum_\ell\overline{\frac{\partial\dot{A}_\ell}{\partial A_\ell}}\right|=k_B\left|\sum_\ell\frac{\partial J_\ell}{\partial A_\ell}\right|=k_B\sum_\ell\{\Gamma^0_{\ell\ell}+\Gamma'_{\ell\ell}\}. \tag{5.8}$$

Namely, the dissipation rate $\Sigma_\ell\Gamma'_{\ell\ell}$ gives the entropy production due to chaos or turbulence.

In the low-frequency case $\tau_M\gg\tau_r$, the fluctuating force $r_\ell(t)$ is specified by

$$\overline{r_\ell(t)}=0, \quad \overline{r_\ell(t)r_n^\dagger(t')}=2\xi_{\ell n}\delta(t-t'), \tag{5.9}$$

where $\xi_{\ell n}$ is the transport coefficients

$$\xi_{\ell n} \equiv \int_0^\infty \langle r_\ell(a,t) r_n^\dagger(a,0) \rangle dt = \sum_m \Gamma'_{\ell m} \langle a_m a_n^\dagger \rangle. \tag{5.10}$$

Then $r_\ell(t)$ may be assumed to be a Gaussian white process. Thus the Markovian equation (5.6) has the structure similar to the linear Langevin equation for Brownian motion. Let $P(a,t)$ be the probability density that $A(t)$ takes a value around a at time t. Then, the stochastic theory of Markovian processes leads to the Fokker-Planck equation [12]

$$\frac{\partial}{\partial t} P(a,t) = \sum_\ell \frac{\partial}{\partial a_\ell} \left[- \sum_n \{ i\Omega^0_{\ell n} - \Gamma^0_{\ell n} - \Gamma'_{\ell n} \} a_n - K_\ell(a) + \sum_n \xi_{\ell n} \frac{\partial}{\partial a_n^\dagger} \right] P(a,t). \tag{5.11}$$

It would be worth noting two remarks here.

(1) Chaos and turbulence in dissipative systems are related to the molecular thermal motions through the energy dissipation as a channel, so that, if the molecular thermal motions vanish as $k_B \to 0$, then the entropy production due to chaos or turbulence also vanishes.

(2) Such energy dissipation due to chaos or turbulence is not explicitly contained in the usual evolution equations (2.7). Therefore, we have to derive an equation, such as Eq. (5.6), which contains the dissipative flux (5.7) explicitly by introducing the renormalization and the coarse graining in time.

VI. CHAOS-INDUCED FRICTION $\gamma'(\omega)$

Let us take the forced damped pendulum (2.1), for which $a_1 = q$, $a_2 = p$, $a_3 = \phi = \omega^0 t + \phi_0$, and q is set to be $-\pi \leqslant q \leqslant \pi$ with mod 2π. Then the three terms of Eq. (2.7) take the form

$$v(a) = \begin{pmatrix} p \\ -\sin q \\ 0 \end{pmatrix}, \quad J^0(a) = \begin{pmatrix} 0 \\ -\gamma^0 p \\ 0 \end{pmatrix},$$

$$K(a) = \begin{pmatrix} 0 \\ b\cos\phi \\ \omega^0 \end{pmatrix}, \tag{6.1}$$

where $\langle q \rangle = \langle \sin q \rangle = 0$, $\langle p \rangle = 0$. Now, applying Eqs. (4.6), (4.15), and (5.4), let us explore the chaos-induced friction and the power spectra of the orbits $a(t)$.

Under time reversal $t \to -t$, $q \to q$, $\omega^0 \to -\omega^0$, we have $p \to -p$, $\phi \to \phi$. Then we obtain $\langle qp \rangle = \langle p\phi \rangle = \langle \phi q \rangle = 0$ so that

$$\langle a_\ell a_m^\dagger \rangle = \langle |a_\ell|^2 \rangle \delta_{\ell m}. \tag{6.2}$$

Since $\langle p \sin q \rangle = \langle \phi \sin q \rangle = 0$, we have

$$i\Omega^0_{\ell n} = \frac{\langle v_\ell(a) a_n \rangle}{\langle |a_n|^2 \rangle} = \begin{pmatrix} 0 & 1 & 0 \\ -\Omega_0^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.3}$$

where $\Omega_0^2 \equiv \langle q \sin q \rangle / \langle q^2 \rangle$. Then Eqs. (4.6) and (4.15) give the linear stochastic equations

$$\dot{q} = p, \quad \dot{\phi} = \omega^0, \tag{6.4}$$

$$\dot{p} = -\Omega_0^2 q - \gamma^0 p - \int_0^t \gamma'(s) p(t-s) ds + r_2(a,t) + b\cos(\omega^0 t + \phi_0), \tag{6.5}$$

where $r_1(a,t) = r_3(a,t) = 0$, $\langle r_2(a,t) p \rangle = 0$,

$$\gamma'(s) \equiv \frac{1}{\langle p^2 \rangle} \langle r_2(a,s) r_2(a,0) \rangle = \gamma'(-s), \tag{6.6}$$

$$r_2(a,t) = -e^{t\mathcal{Q}\Lambda} \{ \sin q - \Omega_0^2 q \} \tag{6.7}$$

with $\mathcal{Q}$ and Λ being

$$\mathcal{Q} f(a) = f(a) - \frac{\langle f(a) q \rangle}{\langle q^2 \rangle} q - \frac{\langle f(a) p \rangle}{\langle p^2 \rangle} p - \langle f(a) \rangle, \tag{6.8}$$

$$\Lambda = p \frac{\partial}{\partial q} - \{ \sin q + \gamma^0 p - b\cos\phi \} \frac{\partial}{\partial p} + \omega^0 \frac{\partial}{\partial \phi}. \tag{6.9}$$

Equation (4.16) gives $\Gamma'_{\ell n}(\omega) = \gamma'(\omega) \delta_{\ell 2} \delta_{n2}$, where

$$\gamma'(\omega) \equiv \frac{1}{\langle p^2 \rangle} \int_0^\infty e^{-i\omega t} \langle r_2(a,t) r_2(a,0) \rangle dt. \tag{6.10}$$

This is the chaos-induced friction coefficient. Then Eq. (4.12) leads to

$$i\omega \Xi_{\ell m}(\omega) - i \sum_n \Omega^0_{\ell n} \Xi_{nm}(\omega) = \delta_{\ell m} - \{ \gamma^0 + \gamma'(\omega) \} \delta_{\ell 2} \Xi_{2m}(\omega). \tag{6.11}$$

Since $i\Omega^0_{\ell n} = \delta_{\ell 1} \delta_{n2} - \Omega_0^2 \delta_{\ell 2} \delta_{n1}$, this leads to

$$i\omega \Xi_{12}(\omega) - \Xi_{22}(\omega) = 0, \tag{6.12}$$

$$i\omega \Xi_{22}(\omega) + \Omega_0^2 \Xi_{12}(\omega) = 1 - \{ \gamma^0 + \gamma'(\omega) \} \Xi_{22}(\omega). \tag{6.13}$$

This is solved to give

$$\Xi_{22}(\omega) \equiv \frac{1}{\langle p^2 \rangle} \int_0^\infty e^{-i\omega t} \langle p(t) p(0) \rangle dt = \frac{\omega}{i(\omega^2 - \Omega_0^2) + \omega \{ \gamma^0 + \gamma'(\omega) \}}, \tag{6.14}$$

and $\Xi_{12}(\omega)=\Xi_{22}(\omega)/i\omega$.

For $a_2(\omega)=p(\omega)$, the power spectrum (5.4) becomes

$$I_p(\omega)=\frac{\langle p^2\rangle}{2\pi}\{\Xi_{22}(\omega)+\text{c.c.}\}, \tag{6.15}$$

$$=\frac{\langle p^2\rangle}{\pi}\frac{\omega^2\{\gamma^0+\text{Re}\,\gamma'(\omega)\}}{\{(\omega^2-\Omega_0^2)+\omega\,\text{Im}\,\gamma'(\omega)\}^2+\omega^2\{\gamma^0+\text{Re}\,\gamma'(\omega)\}^2}, \tag{6.16}$$

where $\omega\neq\pm\omega^0$. For $a_1(\omega)=q(\omega)=p(\omega)/i\omega$,

$$I_q(\omega)=\frac{1}{\omega^2}I_p(\omega). \tag{6.17}$$

Thus it turns out that the structure of the power spectra of the orbits $a(t)$ is determined by the chaos-induced friction coefficient $\gamma'(\omega)$.

In the case of low frequencies $|\omega|\ll 1/\tau_r(\lesssim\gamma^0)$, we may neglect the ω dependence of the friction coefficient $\gamma'(\omega)$. Then the power spectra (6.16) and (6.17) are characterized by Ω_0 and $\gamma=\gamma^0+\gamma'(\omega=0)$ with $\text{Im}\,\gamma'=0$, and become similar to those of the Brownian motion under an elastic force of frequency Ω_0 [12]. Indeed, using Eq. (5.11) with $\xi_{\ell n}=\xi\delta_{\ell 2}\delta_{n2},(\xi=\gamma'\langle p^2\rangle)$ leads to the Kramers equation

$$\frac{\partial}{\partial t}P(q,p,t)=\left[-\frac{\partial}{\partial q}p+\frac{\partial}{\partial p}\{\Omega_0^2q+(\gamma^0+\gamma')p-b\cos(\omega^0t+\phi_0)\}+\xi\frac{\partial^2}{\partial p^2}\right]P(q,p,t). \tag{6.18}$$

Further in the low-frequency case $\tau_M\gg\tau_r$, Eq. (5.8) leads to the entropy production

$$\overline{\dot{S}}=k_B\left|\sum_{\ell}\overline{\frac{\partial\dot{A}_\ell}{\partial A_\ell}}\right|=k_B(\gamma^0+\gamma'), \tag{6.19}$$

where $\gamma'\equiv\gamma'(\omega=0)$. We have $\gamma'\gg\gamma^0$ for a strong chaos.

VII. TURBULENT VISCOSITY $\nu'(k,\omega)$

Let us take $3d$ turbulence in an incompressible fluid governed by the Navier-Stokes equation (2.3), which is assumed to be statistically homogeneous and isotropic.

Then Eqs. (4.6) and (4.15) lead to the linear stochastic equations

$$\dot{u}_{\alpha\mathbf{k}}(t)=-\nu^0k^2u_{\alpha\mathbf{k}}(t)-\int_0^t\nu'(k,s)k^2u_{\alpha\mathbf{k}}(t-s)ds+r_{\alpha\mathbf{k}}(u,t)+K_{\alpha\mathbf{k}}, \tag{7.1}$$

where $\langle r_{\alpha\mathbf{k}}(u,t)u^\dagger_{\beta\mathbf{p}}\rangle=0$,

$$\nu'(k,s)\equiv\frac{\langle r_{\alpha\mathbf{k}}(u,s)r^\dagger_{\alpha\mathbf{k}}(u,0)\rangle}{k^2\langle|u_{\alpha\mathbf{k}}|^2\rangle}, \tag{7.2}$$

$$r_{\alpha\mathbf{k}}(u,t)\equiv e^{t\mathcal{Q}\Lambda}v_{\alpha\mathbf{k}}(u) \tag{7.3}$$

with $\mathcal{Q}$ and Λ being

$$\mathcal{Q}f(u)=f(u)-\sum_\beta\sum_{\mathbf{p}}{}'\frac{\langle f(u)u^\dagger_{\beta\mathbf{p}}\rangle}{\langle|u_{\beta\mathbf{p}}|^2\rangle}u_{\beta\mathbf{p}}, \tag{7.4}$$

$$\Lambda=\sum_\alpha\sum_{\mathbf{k}}{}'\{v_{\alpha\mathbf{k}}(u)-\nu^0k^2u_{\alpha\mathbf{k}}+K_{\alpha\mathbf{k}}\}\frac{\partial}{\partial u_{\alpha\mathbf{k}}}. \tag{7.5}$$

Thus the renormalization of ν^0 by the inertial term $v_{\alpha\mathbf{k}}(u)$ leads to the memory function $\nu'(k,s)$ and the turbulent fluctuating force $r_{\alpha\mathbf{k}}(u,t)$.

Since $\langle r_{\alpha\mathbf{k}}(u,t)u^\dagger_{\alpha\mathbf{k}}\rangle=0$, Eq. (7.1) leads to

$$\Xi_{\alpha\mathbf{k}}(\omega)\equiv\frac{1}{\langle|u_{\alpha\mathbf{k}}|^2\rangle}\int_0^\infty e^{-i\omega t}\langle u_{\alpha\mathbf{k}}(t)u^\dagger_{\alpha\mathbf{k}}(0)\rangle dt$$

$$=\frac{1}{i\omega+\nu^0k^2+\nu'(k,\omega)k^2}, \tag{7.6}$$

where $\nu'(k,\omega)$ is the Laplace transform of Eq. (7.2),

$$\nu'(k,\omega)\equiv\frac{1}{k^2\langle|u_{\alpha\mathbf{k}}|^2\rangle}\int_0^\infty e^{-i\omega t}\langle r_{\alpha\mathbf{k}}(u,t)r^\dagger_{\alpha\mathbf{k}}(u,0)\rangle dt. \tag{7.7}$$

Since $\nu'(k,\omega)k^2$ represents the energy dissipation rate due to turbulence, $\nu'(k,\omega)$ is the turbulent viscosity, and Eq. (7.7) gives the fluctuation-dissipation formula which relates the turbulent viscosity $\nu'(k,\omega)$ to the time-correlation function of the fluctuating force $r_{\alpha\mathbf{k}}$.

In the case of small wave numbers $k\ll k_c$ and low frequencies $|\omega|\ll 1/\tau_r(\lesssim\nu^0k^2)$ where $\tau_M(k)\gg\tau_r(k)$, Eq. (5.8) gives the entropy production

$$\overline{\dot{S}}=3k_B\sum_{\mathbf{k}}{}''\{\nu^0+\nu'(k)\}k^2, \tag{7.8}$$

where $\nu'(k)\equiv\nu'(k,\omega=0)$, and $\Sigma''_{\mathbf{k}}$ is the sum over **k**'s with a cutoff $k_L(\ll k_c)$.

Next let us take isotropic, fully developed turbulence and consider the k dependence of the turbulent viscosity $\nu'(k,\omega)$ in the inertial subrange, where we can neglect the molecular viscosity term and the external force so that the Navier-Stokes equation (2.3) is invariant under a scale transformation [21,22]. Let us assume that the characteristic turnover time t_k of eddies of size $\ell=1/k$ is estimated as

$$t_k\sim\frac{\ell}{\Delta u(\ell)}\sim k^{-z}, \tag{7.9}$$

with an exponent z, where $\Delta u(\ell)$ is the characteristic turnover velocity of the eddies. The linear stochastic equation

(7.1) which is equivalent to the Navier-Stokes equation is also invariant under the scale transformation. This suggests that the following scaling relations hold:

$$u_{\alpha\mathbf{k}}(t)=k^{-\theta}\tilde{u}_{\alpha}\left(\frac{\mathbf{k}}{k},tk^{z}\right), \tag{7.10}$$

$$r_{\alpha\mathbf{k}}(t)=k^{-\theta+z}\tilde{r}_{\alpha}\left(\frac{\mathbf{k}}{k},tk^{z}\right), \tag{7.11}$$

where θ is a certain exponent, and $\tilde{u}_{\alpha}(\mathbf{x},y)$ and $\tilde{r}_{\alpha}(\mathbf{x},y)$ are universal functions of $\mathbf{x}$ and y. If we use the expression $\langle|u_{\alpha\mathbf{k}}|^2\rangle\sim k^{-3-\zeta(2)}$, then Eq. (7.10) leads to $\theta=\frac{1}{2}[3+\zeta(2)]$. Since the average kinetic energy of the eddies is estimated as [21,22]

$$\langle[\Delta u(\ell)]^2\rangle\sim k^{-\zeta(2)} \tag{7.12}$$

with $\zeta(2)=\frac{2}{3}-\mu_{2/3}$ in terms of the intermittency exponent μ_q of order q, Eq. (7.9) leads to $z=1-\frac{1}{2}\zeta(2)$. Putting $\mu\equiv\mu_2$, we have $\mu_{2/3}=-\mu/9$ for the log-normal theory and $\mu_{2/3}=-\mu/3$ for the β model [21]. Therefore we have

$$z=\frac{2}{3}+\frac{1}{2}\mu_{2/3}. \tag{7.13}$$

Substituting Eqs. (7.10) and (7.11) into Eq. (7.7), we obtain

$$\nu'(k,\omega)=k^{-2+2\theta}\int_0^\infty e^{-i\omega t}k^{-2\theta+2z}g(tk^z)dt,$$
$$=k^{-2+z}f(\omega k^{-z}), \tag{7.14}$$

where $f(x)$ is a unique function of x, being independent of the coordinate index α because of the isotropic nature of turbulence. This leads to $\nu'(k,\omega)=k^{-\beta}f(\omega k^{-z})$ with

$$\beta\equiv 2-z=\frac{4}{3}+\frac{1}{2}|\mu_{2/3}|. \tag{7.15}$$

If one neglects the intermittency correction $\mu_{2/3}$ in $\zeta(2)$, then this agrees with the classical result $\beta=\frac{4}{3}$ [8,9].

It would be worth mentioning the renormalization-group theory of turbulence which has been developed first by Forster, Nelson, and Stephen for the Navier-Stokes equation [23]. In this theory the molecular viscosity ν^0 is renormalized by the hydrodynamic modes $u_{\alpha\mathbf{k}}$ by eliminating the modes $u_{\alpha\mathbf{k}}$ with $K\geqslant k>K'=K/e^{\ell}$ $(\ell>1)$ successively starting from $K=k_c$ and deriving a renormalized equation for $k\ll k_c$. This has been done approximately by means of a diagrammatic perturbation theory. In order to obtain the turbulent viscosity, however, the elimination of the hydrodynamic modes is not necessary. Indeed, the renormalization of ν^0 by the nonlinear interactions $v_{\alpha\mathbf{k}}(u)$ is sufficient, and this has been shown exactly in Sec. IV, leading to the linear stochastic equation (7.1) and the turbulent viscosity (7.7).

VIII. EXTERNAL NOISE EFFECT AND NONLINEAR DISSIPATIVE TERM $J^0(A)$

It often occurs that an external noise $R_\ell(t)$ is added so that the evolution equation (2.7) becomes stochastic:

$$\dot{A}_\ell(t)=S_\ell(A(t))+R_\ell(t), \tag{8.1}$$

where the systematic part consists of three terms

$$S_\ell(A)\equiv v_\ell(A)+J^0_\ell(A)+K_\ell(A). \tag{8.2}$$

In the following, we further assume that the dissipative term $J^0_\ell(A)$ is generally nonlinear in contrast to Eq. (2.8), and then we will show how the foregoing theory must be extended.

The external noise $R_\ell(t)$ is assumed to be a Gaussian white process, being specified by

$$\langle R_\ell(t);b\rangle=0,\quad \langle R_\ell(t)R^\dagger_m(t');b\rangle=2D_{\ell m}\delta(t-t'), \tag{8.3}$$

where $\langle\cdots;b\rangle$ denotes the conditional average with the value of A being fixed b at $t=0$:

$$\langle G(t);b\rangle\equiv\frac{\overline{G(t)g_b(0)}}{\overline{g_b(0)}}, \tag{8.4a}$$

$$\overline{G(t)g_b(0)}\equiv\lim_{T\to\infty}\frac{1}{T}\int_0^T G(t+s)g_b(s)ds \tag{8.4b}$$

with $g_b(s)\equiv\delta[A(s)-b]$. The conditional probability density that, given a value b at the initial time $t=0$, one finds $A(t)$ in the range $(a,a+da)$ at a later time t, is given by

$$P(a,t|b,0)=\langle g_a(t);b\rangle. \tag{8.5}$$

Then it is well known that the stochastic theory of Markovian processes leads to

$$\frac{\partial}{\partial t}P(a,t|b,0)=\left[-\sum_\ell\frac{\partial}{\partial a_\ell}S_\ell(a)+\sum_\ell\sum_m\frac{\partial^2}{\partial a_\ell\partial a^\dagger_m}D_{\ell m}\right]P(a,t|b,0)$$
$$\equiv MP(a,t|b,0), \tag{8.6}$$

with the Fokker-Planck operator M [12]. The constant $D_{\ell m}(=D^\dagger_{m\ell})$ represents the intensity of the external noise $R_\ell(t)$.

Since $A_\ell(t)=\int a_\ell g_a(t)da$, the stochastic equation (8.1) leads to

$$\frac{\partial}{\partial t}g_a(t)=Mg_a(t)+F_a(t), \tag{8.7}$$

where $F_a(t)$ is the master fluctuating force

$$F_a(t) \equiv -\sum_{\ell} \frac{\partial}{\partial a_\ell}\{R_\ell(t)\delta(A(0)-a)\}, \tag{8.8}$$

satisfying

$$\langle F_a(t);b\rangle = 0, \quad \int a_\ell F_a(t)da = R_\ell(t). \tag{8.9}$$

This will be shown in the Appendix. Indeed, the Fokker-Planck equation (8.6) is derived from Eq. (8.7) by taking the conditional average $\langle \cdots ;b\rangle$ and using Eqs. (8.8) and (8.9). Equation (8.1) is also derived from Eq. (8.7). Thus any information about the dynamics of $A(t)$ can be derived from Eq. (8.7). Hence Eq. (8.7) is called the *master equation.*

Multiplying the 1st equation of Eq. (8.9) by a function $f(b)$ and integrating over b, we obtain $\overline{F_a(t)f(A(0))}=0$ $=\overline{R_\ell(t)f(A(0))}$. Namely, the external noises $R_\ell(t)$ and $F_a(t)$ are orthogonal to any function of $A(0)$ in contrast to that in Sec. IV noise is orthogonal only to a linear function of $A(0)$. This is because the systematic part $S_\ell(a)$ is nonlinear in contrast to Eq. (4.6).

Integrating Eq. (8.7), and substituting its result into $A_\ell(t)=\int a_\ell g_a(t)da$ yield

$$A_\ell(t) = \int a_\ell(t)g_a(0)da + \int_0^t \int a_\ell(t-s)F_a(s)dads, \tag{8.10}$$

where $a_\ell(t)\equiv e^{t\Lambda}a_\ell$ similarly to Eq. (3.10) but with

$$\Lambda \equiv \sum_\ell S_\ell(a)\frac{\partial}{\partial a_\ell} + \sum_\ell \sum_m D_{\ell m}\frac{\partial^2}{\partial a_\ell \partial a_m^\dagger}. \tag{8.11}$$

It should be noted that the time evolution of $a_\ell(t)$ is deterministic, although that of $A_\ell(t)$ is stochastic due to the presence of the second term of Eq. (8.10).

Let us consider the time-correlation functions (3.1). Since $\overline{F_a(s)A_m^\dagger(0)}=0$, substituting Eq. (8.10) into Eq. (3.1) gives

$$\overline{A_\ell(t)A_m^\dagger(0)} = \langle a_\ell(t)a_m^\dagger\rangle \equiv \int P_*(a)a_\ell(t)a_m^\dagger da \tag{8.12}$$

similarly to Eq. (3.13), where $P_*(a)\equiv\overline{g_a(s)}$ $=\lim_{t\to\infty}P(a,t|b,0)$ is the steady probability density and satisfies $MP_*(a)=0$.

Since $\Lambda a_\ell = S_\ell(a)$, using the projection operator (4.1) leads to

$$\dot{a}(t) = \{i\Omega^0 - \Gamma^0\}\cdot a(t) + e^{t\Lambda}\mathcal{Q}\{v(a)+J^0(a)\} + K(a(t)), \tag{8.13}$$

where $i\Omega^0$ is given by Eq. (4.4) and

$$\Gamma^0 \equiv -\langle J^0(a)a^\dagger\rangle \cdot \langle aa^\dagger\rangle^{-1}. \tag{8.14}$$

Substituting Eq. (4.5) into Eq. (8.13) gives

$$\dot{a}(t) = \{i\Omega^0 - \Gamma^0\}\cdot a(t) - \int_0^t \Gamma'(s)\cdot a(t-s)ds + r(a,t) + K(a(t)), \tag{8.15}$$

where, as the extension of Eqs. (4.7) and (4.15),

$$r(a,t) \equiv e^{t\mathcal{Q}\Lambda}\mathcal{Q}\{v(a)+J^0(a)\}, \tag{8.16}$$

$$\Gamma'(s) \equiv \langle r(a,s)\tilde{r}^\dagger(a)\rangle \cdot \langle aa^\dagger\rangle^{-1}, \tag{8.17}$$

and the mth component of the vector $\tilde{r}(a)$ is given by

$$\tilde{r}_m(a) \equiv -\mathcal{Q}\frac{1}{P_*(a)}M\{a_m P_*(a)\}, \tag{8.18}$$

$$= \mathcal{Q}\{v_m(a)+J_m^0(a)\} - 2\mathcal{Q}\sum_n D_{mn}\frac{\partial}{\partial a_n^\dagger}\ln P_*(a). \tag{8.19}$$

It should be noted that Eqs. (8.16) and (8.17) reduce to Eqs. (4.7) and (4.15) if and only if $J^0(a)$ is linear and there is no external noise $D_{\ell m}=0$.

Since $\langle r(a,t)a^\dagger\rangle = 0$, Eq. (8.15) gives a linear transport equation for the time-correlation matrix $\langle a(t)a^\dagger\rangle$, so that Eq. (8.12) leads to

$$\Xi(\omega) \equiv \int_0^\infty e^{-i\omega t}\overline{A(t)A^\dagger(0)}\cdot[\overline{AA^\dagger}]^{-1}dt = \frac{1}{i(\omega-\Omega^0)+\Gamma^0+\Gamma'(\omega)} \tag{8.20}$$

with the energy dissipation rate

$$\Gamma'(\omega) = \int_0^\infty e^{-i\omega t}\langle r(a,t)\tilde{r}^\dagger(a)\rangle\cdot\langle aa^\dagger\rangle^{-1}dt. \tag{8.21}$$

This is the extension of the fluctuation-dissipation formula (4.16), and shows how the external noises and nonlinearity of $J^0(a)$ modify the transport and energy dissipation.

Taking $\dot{A}_\ell(t)$ of Eq. (8.10) and using Eq. (8.15) for $\dot{a}_\ell(t)$, we obtain the linear stochastic equation

$$\dot{A}(t) = \{i\Omega^0 - \Gamma^0\}\cdot A(t) - \int_0^t \Gamma'(s)\cdot A(t-s)ds + f(t) + K(A(t)), \tag{8.22}$$

where $f(t)$ is the renormalized fluctuating force

$$f(t) \equiv R(t) + \int r(a,t)g_a(0)da + \int_0^t \int r(a,t-s)F_a(s)dads. \tag{8.23}$$

Equation (8.22) is equivalent to Eq. (8.1) but contains the memory function $\Gamma'(s)$ and chaotic or turbulent fluctuating force $r(a,t)$ explicitly in contrast to Eq. (8.1).

IX. SHORT SUMMARY

The particular intention of this paper has been to derive the linear stochastic equations (4.6) for chaos or turbulence from the nonlinear evolution equations (2.7), where the Laplace transform $\Gamma'(\omega)$ of the memory function (4.8) gives the chaotic or turbulent transport coefficients and their energy dissipation rates. This has been done by using the projection-operator method which transforms the nonlinear term $\mathcal{Q}v(a)$ of Eq. (2.7) into the sum of a linear transport term and a nonlinear fluctuating force $r(a,t)$. Indeed this amounts to the renormalization of the molecular transport term $J^0(a)$ by the nonlinear term $\mathcal{Q}v(a)$. Thus the nonlinear evolution equations (2.7) have been transformed into the linear stochastic equations (4.6) which are useful for constructing a stochastic description of chaos and turbulence.

Thus it has turned out that chaos and turbulence bring about various transport processes whose dissipation rates $\Gamma'(\omega)$ are given by the fluctuaton-dissipaton formula (4.16) in terms of the chaotic or turbulent fluctuating forces $r(a,t)$. For the low frequency components of macrovariables $A_\ell(\omega)$ with $|\omega|\ll 1/\tau_r$, the dissipation rates $\Gamma'(\omega)$ may be regarded as constants $\Gamma'=\Gamma'(\omega=0)$. Then the coarse graining of macrovariables in time, given by Eq. (5.5), has enabled us to introduce the entropy production $\overline{\dot{S}}$ due to chaos or turbulence explicitly, as shown in Eq. (5.8).

Then the Laplace transform of the time-correlation functions of macrovariables (4.11) and the power spectra (5.4) can be written in terms of the dissipation rates $\Gamma'(\omega)$, indicating their physical structures explicitly, as shown in the case of forced damped pendulum. The fluctuation-dissipation formula (4.16) for the dissipation rates $\Gamma'(\omega)$ give exact expressions for the chaotic or turbulent transport coefficients. This has been applied to the chaos-induced friction coefficient $\gamma'(\omega)$ in a forced damped pendulum and the turbulent viscosity $\nu'(k,\omega)$ in fully developed turbulence.

If an external noise $R_\ell(t)$ is added as in Eq. (8.1), then it has turned out that Eqs. (4.16) and (4.6) are extended to the fluctuation-dissipation formula (8.21) and the linear stochastic equation (8.22), respectively. Thus it turns out that the stochastic motion of chaos and turbulence can be treated by extending the concept of the fluctuating forces of the Brownian motion, and the linear stochastic equations (4.6) and (8.22) which are non-Markovian give the basis for the stochastic approach to chaos and turbulence. This is also useful for clarifying what transport processes are brought about by chaos and turbulence.

APPENDIX: DERIVATION OF EQ. (8.7)

The stochastic equation (8.1) leads to

$$\frac{\partial}{\partial t}g_a(t)=-\sum_\ell \frac{\partial}{\partial a_\ell}[\{S_\ell(a)+R_\ell(t)\}g_a(t)]=\Omega(a,t)g_a(t). \tag{A1}$$

This is integrated to give

$$g_a(t)=\exp_+\left[\int_0^t \Omega(a,s)ds\right]g_a(0)$$
$$=\int g_b(0)U_b(t)\delta(a-b)db, \tag{A2}$$

where we defined

$$U_b(t)\equiv\exp_-\left[\int_0^t L(b,s)ds\right],$$
$$L(b,t)\equiv\sum_\ell \{S_\ell(b)+R_\ell(t)\}\frac{\partial}{\partial b_\ell}. \tag{A3}$$

Equation (A2) leads to

$$\frac{\partial}{\partial t}g_a(t)=\int g_b(0)\left\{\frac{\partial}{\partial t}U_b(t)\right\}\delta(a-b)db. \tag{A4}$$

To treat this, let us introduce the projection operator [17]

$$\mathcal{P}_Z G(t)=\int \langle G(t);b\rangle g_b(0)db, \tag{A5}$$

where $\langle G(t);b\rangle$ is the conditional average (8.4a). Using $\mathcal{Q}_Z\equiv 1-\mathcal{P}_Z$, let us define

$$\tilde{U}_b(t)\equiv\exp_-\left[\int_0^t \mathcal{Q}_Z L(b,s)ds\right]. \tag{A6}$$

This leads to the operator identity

$$U_b(t)=\tilde{U}_b(t)+\int_0^t U_b(s)\mathcal{P}_Z L(b,s)[\tilde{U}_b(s)]^{-1}\tilde{U}_b(t)ds, \tag{A7}$$

which is a generalization of Eq. (4.5). Therefore, substituting $(\partial/\partial t)U_b(t)=U_b(t)\mathcal{P}_Z L(b,t)+U_b(t)\mathcal{Q}_Z L(b,t)$ into Eq. (A4) and using Eq. (A7) for the second term, we obtain

$$\frac{\partial}{\partial t}g_a(t)=\int g_b(0)\Bigg[U_b(t)\mathcal{P}_Z L(b,t)+\tilde{U}_b(t)\mathcal{Q}_Z L(b,t)$$
$$+\int_0^t U_b(s)\mathcal{P}_Z L(b,s)$$
$$\times[\tilde{U}_b(s)]^{-1}\tilde{U}_b(t)\mathcal{Q}_Z L(b,t)ds\Bigg]\delta(a-b)db. \tag{A8}$$

Here, since $\langle R_\ell(t);b\rangle=0$, we obtain

$$\mathcal{P}_Z L(b,t)\delta(a-b)=-\sum_\ell \frac{\partial}{\partial a_\ell}[S_\ell(a)\delta(a-b)], \tag{A9}$$

$$\mathcal{Q}_Z L(b,t)\delta(a-b)=-\sum_\ell \frac{\partial}{\partial a_\ell}[R_\ell(t)\delta(a-b)]. \tag{A10}$$

The three terms of Eq. (A8) can be transformed into the three terms of Eq. (8.7) as follows. Indeed the first term of Eq. (A8) can be written as

$$-\sum_{\ell} \frac{\partial}{\partial a_{\ell}}[S_{\ell}(a)g_a(t)], \tag{A11}$$

where use has been made of Eq. (A2). The second term of Eq. (A8) takes the form

$$-\sum_{\ell} \frac{\partial}{\partial a_{\ell}}\left[\int g_b(0)\tilde{U}_b(t)\{R_{\ell}(t)\delta(a-b)\}db\right]$$
$$\approx -\sum_{\ell} \frac{\partial}{\partial a_{\ell}}\left[\int g_b(0)\delta(a-b)\{\tilde{U}_b(t)R_{\ell}(t)\}db\right]$$
$$=-\sum_{\ell} \frac{\partial}{\partial a_{\ell}}[R_{\ell}(t)g_a(0)]=F_a(t). \tag{A12}$$

The third term of Eq. (A8) can be written as

$$\sum_{\ell}\sum_{m} \frac{\partial}{\partial a_{\ell}}\frac{\partial}{\partial a_m^{\dagger}}\int\int_0^t g_b(0)U_b(s)\mathcal{P}_Z R_m^{\dagger}(s)$$
$$\times[\tilde{U}_b(s)]^{-1}R_{\ell}(t)\delta(a-b)ds\,db,$$
$$\approx\sum_{\ell}\sum_{m}\frac{\partial^2}{\partial a_{\ell}\partial a_m^{\dagger}}\int\int_0^t g_b(0)U_b(s)\mathcal{P}_Z R_m^{\dagger}(s)R_{\ell}(t)$$
$$\times\delta(a-b)ds\,db,$$
$$=\sum_{\ell}\sum_{m}\frac{\partial^2}{\partial a_{\ell}\partial a_m^{\dagger}}[D_{\ell m}g_a(t)], \tag{A13}$$

where use has been made of Eqs. (8.3) and (A2). The substitution of Eqs. (A11)–(A13) into Eq. (A8) gives Eq. (8.7).

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